MACAULAY STYLE FORMULAS FOR TORIC RESIDUES

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Abstract. We present an explicit formula for computing toric residues as a quotient of two determinants, á la Macaulay, where the numerator is a minor of the denominator. We also give an irreducible representation of toric residues by extending the theory of subresultants to monomials of critical degree in the homogeneous coordinate ring of the corresponding toric variety.

1. introduction

The toric residue of $n+1$ divisors on an n-dimensional toric variety was first introduced by Cox [Cox2] in the case when all divisors are ample of the same class, and extended to the general case by Cattani, Cox, and Dickenstein [CCD]. Toric residues have been found to be useful in a variety of contexts such as mirror symmetry [BM], the Hodge structure of hypersurfaces [BC], and in the study of sparse resultants [CDS2].

Another related, perhaps more familiar, notion is the global residue in the torus. Given a system $f_1, \ldots, f_n$ of n Laurent polynomials in n variables with a finite set of common zeroes in the torus $T = (C^*)^n$, and another Laurent polynomial q, the global residue of q with respect to $f_1, \ldots, f_n$ is the sum of the Grothendieck residues of the differential form

$$\phi_q = \frac{q}{f_1 \cdots f_n} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n},$$

at each zero of the f_i. This turns out to be a rational function in the coefficients in the f_i and has a wide variety of applications in algebra and analysis. The residue in the torus has been studied by Khovanskii, Gelfond, and Soprounov [Kho, GK, Sop]. Cattani and Dickenstein [CD, Theorem 4] showed that the global residue in the torus is equal to a particular toric residue in the sense above of [Cox2, CCD]. The precise relationship between the toric residue and the global residue in the torus is discussed in Section 6.

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In this paper, we present an explicit formula for computing a toric residue as a quotient of two determinants, where the numerator is a minor of the denominator. This is an improvement over earlier algorithms in [CCD, CD] by eliminating costly Gröbner basis computations.

Both the numerator and denominator of our quotient formula turn out to be divisible by the same extraneous factor. It would be useful to have a description of the residue in reduced form. Indeed, the denominator of the toric residue has already been identified with the sparse resultant [CDS2]. Our second main result is an identification of the numerator with a "toric subresultant", analagous to the multivariate subresultant of Chardin [Cha1]. In the dense case, this numerator and its properties have been deeply studied by Jouanolou in [J1, J2, J3]. Our results may be regarded as a generalization of Jouanolou's work.

We start with some notation on toric varieties. For more details we refer the reader to [Ful, Cox1, Cox2]. Let X be a projective toric variety of dimension n, hence determined by a fan $\Sigma \subset \mathbb{R}^n$. The generators of the 1-dimensional cones in Σ will be denoted $\eta_0, \ldots, \eta_{s-1}$. The Chow group $A_{n-1}(X)$ has rank $s-n$. We work in the polynomial ring $S := \mathbb{A}[x_0, \ldots, x_{s-1}]$ where each variable x_i corresponds to ray η_i and hence to a torus-invariant divisor D_i of X. We grade S by declaring that the monomial $\prod x_i^{a_i}$ has degree $[\sum a_i D_i] \in A_{n-1}(X)$. The base ring $\mathbb{A}$ will be the coefficient space of our polynomial system and is specified below. We abbreviate $\beta_0 := [\sum_i D_i]$, the anticanonical class on X. The *irrelevant ideal* $B(\Sigma)$ is generated by the elements $\hat{x}_\sigma = \prod_{\eta_i \notin \sigma} x_i$ where σ ranges over all n-dimensional cones in Σ.

We now recall the definition of the toric residue from [Cox2]. This depends on $n+1$ generic S-homogeneous polynomials which are homogenizations of Laurent polynomials in n variables. Given a polynomial of a certain critical degree we construct a differential n-form which gives rise to a top cohomology class of the canonical sheaf of differentials $\Omega^n_X = \mathcal{O}(-\beta_0)$. The toric residue is defined to be the trace of this cohomology class.

Formally, pick ample degrees $\alpha_0, \ldots, \alpha_n$ and consider generic polynomials:

$$F_i(u,x) := \sum_{a \in \mathcal{A}_i} u_{ia} x^a, \ i = 0, \ldots, n. \tag{1}$$

where $\mathcal{A}_i := \{a \in \mathbb{N}^s : \deg(a) = \alpha_i\}$. Set $\mathbb{A} := \mathbb{Q}[u_{ia}; i = 0, \ldots, n; a \in \mathcal{A}_i]$, and write $Q(\mathbb{A})$ for the field of quotients of $\mathbb{A}$. Let $\rho = \sum_i \alpha_i - \beta_0$, which is the *critical degree* of our system. For any subset $I =$

$\{i_1, \ldots, i_n\}$ of $\{0, \ldots, s-1\}$ we write

$$\det(\eta_I) := \det(\langle e_l, \eta_{i_j}\rangle_{1\le l,j\le n}), \quad dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_n}, \quad \hat{x_I} = \prod_{j\notin I} x_j.$$

The *Euler form* on X is the sum over all n-element subsets I of $\{1, \ldots, s\}$:

$$\Omega := \sum_{|I|=n} \det(\eta_I)\hat{x_I} dx_I.$$

The polynomials F_i determine an open cover $U_i = \{x \in X \;:\; F_i(x) \neq 0\}$. A polynomial $g \in S_\rho$ gives an element $\omega_g = \frac{g\Omega}{F_0\cdots F_n}$ which is a Čech cocycle of degree n with respect to the open cover U_i. Therefore, there is an induced cohomology class $[\omega_g] \in H^n(X, \Omega^n_X)$, and we define $\mathbf{Residue}_F(g) = \mathrm{Tr}_X([\omega_g])$. The toric residue is thus a map $\mathbf{Residue}_F : S_\rho \to Q(\mathbb{A})$.

The remainder of this paper is organized as follows: Section 2 gives a combinatorial construction of explicit elements of S_ρ with residue ± 1, playing the role of the toric Jacobian from [CDS2] and generalizing the elements Δ_σ constructed in [CCD] on simplicial toric varieties. This is used in Section 3 to give the promised Macaulay style residue formula.

Sections 4 and 5 discuss the reduced numerator of the toric residue of a monomial h. This is shown to be a toric subresultant associated to the systen $F_0, \ldots, F_n$ and the monomial h. In the dense case, the subresultant was introduced by [Cha1], and different algorithms for computing subresultants have been developed in [Cha2, Sza].

In Section 6, we will show how our results can be used to compute global residues in the torus. We see how our results generalize some explicit formulas given by Macaulay in [Mac] for computing global residues of dense homogeneous systems.

2. Elements with nonzero residue

Let X be a projective, but not necessarily simplicial, toric variety with fan Σ and $F_0, \ldots, F_n$ generic elements of ample degrees α_i as above.

Pick a complete flag $\bar\sigma = \{0\} \subset \sigma_1 \subset \cdots \subset \sigma_n$ where each σ_i is a cone of dimension i in Σ. For $i = 1, \ldots, n$, let z_i be the product of all variables x_j such that $\eta_j \in \sigma_i$ but $\eta_j \notin \sigma_{i-1}$. We set $z_{n+1} = \prod_{\eta_j\notin\sigma_n} x_j$.

We will see that each F_j can be written in the form

$$F_j = \sum_{i=1}^{n+1} A_{ij} z_i.$$

The $(n+1)\times(n+1)$-determinant $\Delta_{\bar{\sigma}} = \det(A_{ij})$ is in S_ρ. We have the following theorem generalizing [CCD, Theorem 0.2].

Theorem 2.1. *Suppose X, F, and $\bar{\sigma}$ are as above. Then* $\mathbf{Residue}_F\,(\Delta_{\bar{\sigma}}) = \pm 1$.

Note that if X were simplicial then each of $z_1,\dots,z_n$ would be a single variable corresponding to the generators of the cone σ_n and z_{n+1} would be the product of all of the remaining variables. The element $\Delta_{\bar{\sigma}}$, in this case, is the same as the element Δ_{σ_n} from [CCD].

Proof. The η_i are ample classes, therefore $\langle F_0,\dots,F_n\rangle \subset B(\Sigma)$. We next show that $B(\Sigma)\subset\langle z_1,\dots z_{n+1}\rangle$. Consider a generator $\hat{x}_\tau$ where τ is a maximal cone of Σ. Recall that $\hat{x}_\tau = \prod_{\eta_i\notin\tau} x_i$. Now $\tau\cap\sigma_n$ is a face of σ_n. Choose i such that $\tau\cap\sigma_n\supset\sigma_{i-1}$ but $\tau\cap\sigma_n\not\subset\sigma_i$. Now, we see that the one dimensional cones in $\tau\cap\sigma_i$ are all contained in σ_{i-1}, and so none of the one dimensional cones in z_i are in τ. Therefore z_i divides $\hat{x}_\tau$ as desired. Hence we can write (nonuniquely) for $j = 0,\dots n$

$$F_j = \sum_{i=1}^{n+1} A_{ij} z_i.$$

Now an application of the global transformation law (Theorem 0.1 in [CCD]), shows that $\mathbf{Residue}_F\,(\Delta_{\bar{\sigma}}) = \mathbf{Residue}_z\,(1)$. And so we need only prove that $\mathbf{Residue}_z\,(1) = \pm 1$.

Let X' be a new toric variety arising from a simplicial refinement Σ' of Σ with the same 1-dimensional cones, hence the same coordinate ring, albeit with a smaller irrelevant ideal $B(\Sigma')\subset B(\Sigma)$. We have a natural map $f: X'\to X$. Let $\mathcal{U}$ be the open cover on X defined by $\{U_i = \{x\in X \;:\; z_i\neq 0\}\}$. We have an analagous collection of open sets $\mathcal{U}'$ on X' defined by the same equations $\{U_i' = \{x\in X' \;:\; z_i\neq 0\}\}$. As $B(\Sigma')\subset B(\Sigma)\subset\langle z_1,\dots z_{n+1}\rangle$, $\mathcal{U}'$ is also an open cover of X'.

Now, on X, the element $1\in S$ gives the n-form

$$\omega = \frac{\Omega}{z_1\cdots z_{n+1}} = \frac{\Omega}{x_0\cdots x_{s-1}}. \tag{2}$$

This form is defined on the open set

$$U = \{x\in X \;:\; z_1(x)\neq 0,\dots,z_{n+1}(x)\neq 0\},$$

which is the same as the open set defined by $x_0\neq 0,\dots,x_{s-1}\neq 0$. The map f defines an isomorphism on this open set, and we can pull back ω to get a new form $\omega' = f^*(\omega)$ on $f^{-1}(U)\subset X'$, defined by the same formula (2). Now, $f^{-1}\mathcal{U}$ is an open cover of X', and moreover it refines the open cover $\mathcal{U}'$, since $z_i\neq 0$ on each $f^{-1}(U_i)$. Since the natural

map from Čech cohomology to sheaf cohomology respects refinement $[\omega'_{\mathcal{U}'}] = [\omega'_{f^{-1}\mathcal{U}}]$ as cohomology classes in $H^n(X', \Omega'^n_X)$. Moreover, f is birational so $Tr'_X \circ f^* = Tr_X$. Putting it together we have:

$$\textbf{Residue}_{\,z}(1)_X = \pm 1 \iff \textbf{Residue}_{\,z}(1)_{X'} = \pm 1. \tag{3}$$

In [CCD] it was shown that $Tr([\omega'_{\mathcal{V}}]) = \pm 1$ where $\mathcal{V}$ is an open cover coming from the simplicial construction of $\Delta_{\sigma'}$, and ω' is the same n-form as in (2). So, we need only show that the cohomology classes $[\omega'_{\mathcal{U}'}]$ and $[\omega'_{\mathcal{V}}]$ coincide for some top dimensional cone σ' giving rise to the cover $\mathcal{V}$. This will be done by picking the appropriate σ', and thereby $\mathcal{V}$, and finding an open cover $\mathcal{W}$ refining both $\mathcal{U}'$ and $\mathcal{V}$ for which ω' still determines a Čech cocycle.

Lemma 2.2. *There is a unique (simplicial) cone $\sigma' \in \Sigma'$ of dimension n generated by $\{\eta_{i_1}, \dots \eta_{i_n}\}$ such that the corresponding variables $x_{i_1}, \dots, x_{i_n}$ satisy $x_{i_k} \mid z_k$ and moreover $z_1 = \cdots = z_n = 0$ on X' if and only if $x_{i_1} = \cdots = x_{i_n} = 0$.*

Proof. We proceed by induction to show that $z_1 = \cdots = z_k = 0$ if and only if $x_{i_1} = \cdots = x_{i_k} = 0$ for a unique k dimensional cone $\sigma'_k \in \Sigma'$. The base case $k = 1$ is trivial as z_1 is a single variable x_{i_1} corresponding to a unique 1 dimensional cone. For the inductive step suppose $x_1 = \cdots = x_{i_k} = 0$ and $z_{k+1} = 0$. Let σ'_k be the cone corresponding to $x_{i_1}, \dots, x_{i_k}$. The torus orbits of X' are in one to one correspondance with the cones in Σ'. The orbit corresponding to a cone σ' is specified exactly by the vanishing of the variables corresponding to the 1-dimensional generators. In particular if some variable $x_{i_{k+1}}$ dividing z_{k+1} vanishes, then there is a cone in Σ' containing the rays corresponding to $x_{i_1}, \dots, x_{i_{k+1}}$. As Σ' is simplicial, the above set of rays must itself be a $(k+1)$-dimensional cone in Σ'. So, it is enough to show that there is a unique such cone.

Recall that we have a complete flag of cones $\sigma_0 \subset \cdots \subset \sigma_k \subset \cdots \sigma_n \subset \Sigma$. By construction, $\sigma'_k \subset \sigma_k$ and z_{k+1}, is made up of the rays in σ_{k+1} but not in σ_k. Now σ_k was a facet of σ_{k+1} and σ'_k is a cone in its triangulation in Σ'. Therefore, there is a unique cone σ'_{k+1} in the triangulation of σ_{k+1} containing σ'_k (No two cones in a triangulation can meet in a facet of the original cone). Let $x_{i_{k+1}}$ be the additional generator of σ'_{k+1}. This completes the induction and we take $\sigma' = \sigma'_n$. □

Now, let $z'_k = x_{i_k}$ for $k = 1, \ldots, n$, and $z'_{n+1} = \prod_{\eta_j \notin \sigma'} x_j$ the product of the remaining variables. We define the open cover $\mathcal{V}$ by

$$V_i = \{x \in X' \ : \ z'_i \neq 0\}.$$

By the above lemma we have $U'_i \subset V_i$ for $i = 1, \ldots, n$, $V'_{i+1} \subset U'_{i+1}$, and $\bigcup_{i=1}^n V_i = \bigcup_{i=1}^n U'_i$. Therefore the open cover $\mathcal{W} = \{U'_1, \ldots, U'_n, V_{n+1}\}$ refines both $\mathcal{U}'$ and $\mathcal{V}$, and both $\omega'_{\mathcal{U}'}$ and $\omega'_{\mathcal{V}}$ map to a well defined Čech cocycle $\omega'_{\mathcal{W}}$. Hence $\omega'_{\mathcal{U}'}$ and $\omega'_{\mathcal{V}}$ induce the same cohomology class, and hence have the same trace, as desired. So $\mathbf{Residue}_x(1)_{X'} = \pm 1$ which completes the proof by virtue of (3). □

Example 2.3. *Let $O \subset \mathbb{R}^3$ be the octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. This determines a toric variety X_O whose normal fan has generators $\eta_0 = (-1, -1, -1)$, $\eta_1 = (-1, -1, 1)$, $\eta_2 = (-1, 1, -1)$, $\ldots$, $\eta_7 = (1, 1, 1)$ and top dimensional cones determined by the spans of the sets of four rays with one coordinate fixed. Pick the ample degrees $\alpha_0 = \cdots = \alpha_3 = \sum_{i=0}^7 D_i$ and consider the generic system:*

$$\begin{array}{rcl}
F_0 & := & a_0x_0^2x_1^2x_2^2x_3^2 + a_1x_0^2x_1^2x_4^2x_5^2 + a_2x_0^2x_2^2x_4^2x_6^2 + a_3x_0x_1x_2x_3x_4x_5x_6x_7 \\
& + & a_4x_1^2x_3^2x_5^2x_7^2 + a_5x_2^2x_3^2x_6^2x_7^2 + a_6x_4^2x_5^2x_6^2x_7^2 \\
F_1 & := & b_0x_0^2x_1^2x_2^2x_3^2 + b_1x_0^2x_1^2x_4^2x_5^2 + b_2x_0^2x_2^2x_4^2x_6^2 + b_3x_0x_1x_2x_3x_4x_5x_6x_7 \\
& + & b_4x_1^2x_3^2x_5^2x_7^2 + b_5x_2^2x_3^2x_6^2x_7^2 + b_6x_4^2x_5^2x_6^2x_7^2 \\
F_2 & := & c_0x_0^2x_1^2x_2^2x_3^2 + c_1x_0^2x_1^2x_4^2x_5^2 + c_2x_0^2x_2^2x_4^2x_6^2 + c_3x_0x_1x_2x_3x_4x_5x_6x_7 \\
& + & c_4x_1^2x_3^2x_5^2x_7^2 + c_5x_2^2x_3^2x_6^2x_7^2 + c_6x_4^2x_5^2x_6^2x_7^2 \\
F_3 & := & d_0x_0^2x_1^2x_2^2x_3^2 + d_1x_0^2x_1^2x_4^2x_5^2 + d_2x_0^2x_2^2x_4^2x_6^2 + d_3x_0x_1x_2x_3x_4x_5x_6x_7 \\
& + & d_4x_1^2x_3^2x_5^2x_7^2 + d_5x_2^2x_3^2x_6^2x_7^2 + d_6x_4^2x_5^2x_6^2x_7^2
\end{array}$$

As our complete flag $\bar{\sigma}$ we pick $\{0\} \subset \{\eta_0\} \subset \{\eta_0, \eta_1\} \subset \{\eta_0, \eta_1, \eta_2, \eta_3\}$. This gives $z_0 = x_0$, $z_1 = x_1$, $z_2 = x_2x_3$, $z_3 = x_4x_5x_6x_7$ and we can write

$$\begin{array}{rcl}
F_0 & = & z_0(a_0x_0x_1^2x_2^2x_3^2 + a_1x_0x_1^2x_4^2x_5^2 + a_2x_0x_2^2x_4^2x_6^2 + a_3x_1x_2x_3x_4x_5x_6x_7) \\
& + & z_1(a_4x_1x_3^2x_5^2x_7^2) + z_2(a_5x_2x_3x_6^2x_7^2) + z_3(a_6x_4x_5x_6x_7),
\end{array}$$

and similarly for F_1, F_2, F_3. Therefore:

$$\begin{array}{rcl}
\Delta_{\bar{\sigma}} & = & \det \begin{pmatrix}
a_0x_0x_1^2x_2^2x_3^2 + \cdots & a_4x_1x_3^2x_5^2x_7^2 & a_5x_2x_3x_6^2x_7^2 & a_6x_4x_5x_6x_7 \\
b_0x_0x_1^2x_2^2x_3^2 + \cdots & b_4x_1x_3^2x_5^2x_7^2 & b_5x_2x_3x_6^2x_7^2 & b_6x_4x_5x_6x_7 \\
c_0x_0x_1^2x_2^2x_3^2 + \cdots & c_4x_1x_3^2x_5^2x_7^2 & c_5x_2x_3x_6^2x_7^2 & c_6x_4x_5x_6x_7 \\
d_0x_0x_1^2x_2^2x_3^2 + \cdots & d_4x_1x_3^2x_5^2x_7^2 & d_5x_2x_3x_6^2x_7^2 & d_6x_4x_5x_6x_7
\end{pmatrix} \\
& = & [0456]x_0x_1^3x_2^3x_3^5x_4x_5^3x_6^3x_7^5 + [1456]x_0x_1^3x_2x_3^3x_4^3x_5^5x_6^3x_7^5 \\
& + & [2456]x_0x_1x_2^3x_3^3x_4^3x_5^3x_6^5x_7^5 + [3456]x_1^2x_2^2x_3^4x_4^2x_5^4x_6^4x_7^6.
\end{array}$$

Here the "bracket" $[0456]$ *denotes the* 4×4 *determinant:*

$$\det \begin{pmatrix} a_0 & a_4 & a_5 & a_6 \\ b_0 & b_4 & b_5 & b_6 \\ c_0 & c_4 & c_5 & c_6 \\ d_0 & d_4 & d_5 & d_6 \end{pmatrix}.$$

Remark 2.4. *The construction uses that the* α_i *are ample degrees. It is still an open problem to find an explicit element of nonzero residue in a more general setting, for example when the* α_i *correspond to nef and big divisors, i.e. have* n*-dimensional support. Indeed this is the only obstruction to generalizing the Macaulay style formula of the next section to this more general case.*

3. Macaulay Style Formulas for Residues

We now show how to use the element $\Delta_{\bar\sigma}$ to give an explicit Macaulay formula for the residue. We will need the following result. Consider the map of free $\mathbb{A}$ modules:

$$\begin{array}{rccc} \phi : & S_{\rho-\alpha_0} \oplus \cdots \oplus S_{\rho-\alpha_n} \oplus \mathbb{A} & \to & S_\rho \\ & (G_0, \ldots, G_n, c) & \mapsto & \sum_{i=0}^n G_i F_i + c\,\Delta_{\bar\sigma}. \end{array} \tag{4}$$

Theorem 3.1. *(Codimension 1 theorem) Let* X *be a projective toric variety, and* F *a generic ample system as above, then the map* ϕ *described above is generically surjective. Equivalently, the degree* ρ *component of the quotient* $S_F := S/\langle F_0, \ldots, F_n\rangle$ *has* $Q(\mathbb{A})$ *dimension* 1.

The proof is postponed to the next section. When X is simplicial, this result is due to [CCD]. In a forthcoming paper, Cox and Dickenstein [CoD] prove a much more general codimension theorem which implies the theorem above.

Let $\mathbb{M}$ be the matrix associated with the $\mathbb{A}$-linear map ϕ in the monomial bases. As in [Mac], we shall index the rows of $\mathbb{M}$ in correspondence with the elements of the monomial basis of the domain. Fix a monomial $h \in S_\rho$, and let $\tilde{\mathbb{M}}$ be any square maximal submatrix with nonvanishing determinant. It turns out that one of the rows of $\tilde{\mathbb{M}}$ must be indexed by $(0, \ldots, 0, 1)$. This is due to the fact that $\Delta_{\bar\sigma}$ does not belong to the ideal generated by $F_0, \ldots, F_n$ if this family does not have any common zero in X. Let $\tilde{\mathbb{M}}_h$ be the square submatrix of $\tilde{\mathbb{M}}$ made by deleting the row indexed by $(0, \ldots, 0, 1)$ and the column indexed by h. Now we are ready for the main result of this section.

Theorem 3.2.

$$\textbf{Residue}_F\,(h) = \pm\frac{\det(\tilde{\mathbb{M}}_h)}{\det(\tilde{\mathbb{M}})}.$$

Proof. Let $\tilde{M}^h$ be the matrix $\tilde{\mathbb{M}}$ modified as follows: we multiply by h all the elements in the column indexed by h.

Then, it turns out that $\det(\tilde{M}^h) = h\det(\tilde{\mathbb{M}})$. On the other hand, performing elementary operations in the columns of $\tilde{\mathbb{M}}$ and expanding the determinant along the column indexed by h, it turns out that

$$h\det(\tilde{\mathbb{M}}) = \det(\tilde{M}^h) = G_0F_0 + \cdots + G_nF_n \pm \det(\tilde{\mathbb{M}}_h)\,\Delta_{\bar{\sigma}}.$$

By taking the residue of both sides, we get

$$\textbf{Residue}_F\,(h)\det(\tilde{\mathbb{M}}) = \pm\det(\tilde{\mathbb{M}}_h)\textbf{Residue}_F\,(\Delta_{\bar{\sigma}}) = \pm\det(\tilde{\mathbb{M}}_h)$$

□

Remark 3.3. *In the case of "generalized unmixed systems", when each α_i is an integer multiple of a fixed ample degree α, one can also use the toric Jacobian $J(F)$, scaled by an appropriate constant, in place of the element $\Delta_{\bar{\sigma}}$. The Jacobian has the computational disadvantage of having much larger support than $\Delta_{\bar{\sigma}}$ although it has the advantage of being intrinsic to the toric system and not dependent on a choice of $\bar{\sigma}$.*

The following result is a straightforward consequence of Theorem 3.2, and says that for *any* polynomial P of critical degree, $\textbf{Residue}_F\,(P)$ may be computed as a quotient of two determinants.

Corollary 3.4. *Let $P = \sum_{\deg(a)=\rho} p_a x^a$, where p_a are constants. As before, let $\tilde{\mathbb{M}}$ any square maximal submatrix of $\mathbb{M}$ having nonzero determinant. Let $\tilde{\mathbb{M}}_P$ be the matrix $\tilde{\mathbb{M}}$ modified as follows: for every monomial a of critical degree, we replace the input indexed by $((0,\ldots,0,1);a)$ with the coefficient p_a. Then, it turns out that*

$$\textbf{Residue}_F\,(P) = \pm\frac{\det(\tilde{\mathbb{M}}_P)}{\det(\tilde{\mathbb{M}})}.$$

Proof. Expand $\det(\tilde{\mathbb{M}}_P)$ by the row indexed by $\Delta_{\bar{\sigma}}$, and use the linearity of the residue and Theorem 3.2. □

Example 3.5. *Let $X = \mathbb{P}^1\times\mathbb{P}^1$ whose fan has the following 1-dimensional generators: $\eta_1 = (1,0)$, $\eta_2 = (0,-1)$, $\eta_3 = (-1,0)$ and $\eta_4 = (0,1)$. We pick the ample degrees $\alpha_0 = D_2 + D_3$, $\alpha_1 = 2D_2 + D_3$, $\alpha_3 = D_2 + 2D_3$*

and consider the following generic polynomials having those degrees:

$$\begin{aligned}
F_0 &:= a_0x_2x_3 + a_1x_1x_2 + a_2x_1x_4 + a_3x_3x_4,\\
F_1 &:= b_0x_2^2x_3 + b_1x_1x_2^2 + b_2x_2x_2x_4 + b_3x_1x_2x_4 + b_4x_3x_4^2 + b_5x_1x_4^2,\\
F_2 &:= c_0x_2x_3^2 + c_1x_1x_2x_3 + c_2x_1^2x_2 + c_3x_3^2x_4 + c_4x_1x_3x_4 + c_5x_1^2x_4.
\end{aligned}$$

The critical degree is $-D_1 + 2D_2 + 2D_3 - D_4$ *and may be identified with the set of nine integer points lying in the interior of a* 3×3 *square having integer vertices and edges parallel to the axes (see* [CDS2]*). To compute* $\Delta_{\bar\sigma}$ *we can take* $z_1 = x_1$, $z_2 = x_2$, $z_3 = x_3x_4$.

$$\Delta_z = \det\begin{pmatrix} a_1x_2 + a_2x_4 & a_0x_3 & a_3\\ b_1x_2^2 + b_3x_2x_4 + b_5x_4^2 & b_0x_2x_3 + b_2x_3x_4 & b_4x_4\\ c_1x_2x_3 + c_2x_1x_2 + c_4x_3x_4 + c_5x_1x_4 & c_0x_3^2 & c_3x_3 \end{pmatrix}.$$

In this case, $\mathbb{M}$ *is the following* 9×9 *matrix:*

$$\mathbb{M} = \begin{pmatrix}
a_3 & a_2 & 0 & a_0 & a_1 & 0 & 0 & 0 & 0\\
0 & a_3 & a_2 & 0 & a_0 & a_1 & 0 & 0 & 0\\
0 & 0 & 0 & a_3 & a_2 & 0 & a_0 & a_1 & 0\\
0 & 0 & 0 & 0 & a_3 & a_2 & 0 & a_0 & a_1\\
b_4 & b_5 & 0 & b_2 & b_3 & 0 & b_0 & b_1 & 0\\
0 & b_4 & b_5 & 0 & b_2 & b_3 & 0 & b_0 & b_1\\
0 & 0 & 0 & c_3 & c_4 & c_5 & c_0 & c_1 & c_2\\
c_3 & c_4 & c_5 & c_0 & c_1 & c_2 & 0 & 0 & 0\\
\mathbf{d_1} & \mathbf{d_2} & \mathbf{d_3} & \mathbf{d_4} & \mathbf{d_5} & \mathbf{d_6} & \mathbf{d_7} & \mathbf{d_8} & \mathbf{d_9}
\end{pmatrix},$$

where the $\mathbf{d_k}$ *are the coefficients of* $\Delta_{\bar\sigma}$*:*

$$\begin{aligned}
\mathbf{d_1} &:= -a_2b_4c_0 + b_5a_3c_0 + a_2c_3b_2 + c_4a_0b_4 - c_4a_3b_2 - b_5a_0c_3\\
\mathbf{d_2} &:= c_5a_0b_4 - c_5a_3b_2\\
\mathbf{d_3} &:= 0\\
\mathbf{d_4} &:= a_1c_3b_2 - a_1b_4c_0 + a_2c_3b_0 - b_3a_0c_3 + b_3a_3c_0 + c_1a_0b_4 - c_1a_3b_2 - c_4a_3b_0\\
\mathbf{d_5} &:= c_2a_0b_4 - c_2a_3b_2 - c_5a_3b_0\\
\mathbf{d_6} &:= 0\\
\mathbf{d_7} &:= a_1c_3b_0 - a_0b_1c_3 + a_3b_1c_0 - c_1a_3b_0\\
\mathbf{d_8} &:= 0\\
\mathbf{d_9} &:= 0.
\end{aligned}$$

It turns out that $\det(\mathbb{M})$ *equals the sparse resultant of the* F_i*'s. Let*

$$\begin{aligned}
P := {} & \mathbf{p_1}x_3^2x_4^2 + \mathbf{p_2}x_1x_3x_4^2 + \mathbf{p_3}x_1^2x_4^2 + \mathbf{p_4}x_2x_3^2x_4 + \mathbf{p_5}x_1x_2x_3x_4\\
& + \mathbf{p_6}x_1^2x_2x_4 + \mathbf{p_7}x_2^2x_3^2 + \mathbf{p_8}x_1x_2^2x_3 + \mathbf{p_9}x_1^2x_2^2
\end{aligned}$$

be any polynomial of critical degree. Following the notation of Corollary 3.4, we have that $\tilde{\mathbb{M}} = \mathbb{M}$ *and*

$$\tilde{\mathbb{M}}_P = \begin{pmatrix} a_3 & a_2 & 0 & a_0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & a_3 & a_2 & 0 & a_0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & a_2 & 0 & a_0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & a_3 & a_2 & 0 & a_0 & a_1 \\ b_4 & b_5 & 0 & b_2 & b_3 & 0 & b_0 & b_1 & 0 \\ 0 & b_4 & b_5 & 0 & b_2 & b_3 & 0 & b_0 & b_1 \\ 0 & 0 & 0 & c_3 & c_4 & c_5 & c_0 & c_1 & c_2 \\ c_3 & c_4 & c_5 & c_0 & c_1 & c_2 & 0 & 0 & 0 \\ \mathbf{p_1} & \mathbf{p_2} & \mathbf{p_3} & \mathbf{p_4} & \mathbf{p_5} & \mathbf{p_6} & \mathbf{p_7} & \mathbf{p_8} & \mathbf{p_9} \end{pmatrix}.$$

So, we have that $\mathbf{Residue}_F\,(P) = \pm\frac{\det(\tilde{\mathbb{M}}_P)}{\det(\tilde{\mathbb{M}})}$.

Example 3.6. *Consider the generic system introduced in Example 2.3. In this case,* $\mathbb{M}$ *is a* 101×63 *size matrix. With the aid of Maple, we have found a square maximal minor* $\tilde{\mathbb{M}}$ *by choosing the rows indexed by the element* $\Delta_{\bar{\sigma}}$*, as constructed in Example 2.3, and the following monomials:*

- *In* $S_{\rho-\alpha_0}$:

$x_0^4x_1^4x_2^4x_3^4$, $x_0^4x_1^4x_4^4x_5^4$, $x_0^4x_2^4x_4^4x_6^4$, $x_1^4x_3^4x_5^4x_7^4$, $x_2^4x_3^4x_6^4x_7^4$, $x_4^4x_5^4x_6^4x_7^4$, $x_0^4x_1^4x_2^2x_3^2x_4^2x_5^2$, $x_0^4x_1^2x_2^4x_3^2x_4^2x_6^2$, $x_0^3x_1^3x_2^3x_3^3x_4x_5x_6x_7$, $x_0^2x_1^4x_2^2x_3^4x_5^2x_7^2$, $x_0^2x_1^2x_2^4x_3^4x_6^2x_7^2$, $x_0^2x_1^2x_2^2x_3^2x_4^2x_5^2x_6^2x_7^2$, $x_0^4x_1^2x_4^4x_5^2x_2^2x_6^2$, $x_0^3x_1^3x_4^3x_5^3x_2x_3x_6x_7$, $x_0^2x_1^4x_4^2x_5^4x_3^2x_7^2$, $x_0^2x_1^2x_4^4x_5^4x_6^2x_7^2$, $x_0^3x_2^3x_4^3x_6^3x_1x_3x_5x_7$, $x_0^2x_2^4x_4^2x_6^4x_3^2x_7^2$, $x_0^2x_2^2x_4^4x_6^4x_5^2x_7^2$, $x_0x_1^3x_2x_3^3x_4x_5^3x_6x_7^3$, $x_0x_1x_2^3x_3^3x_4x_5x_6^3x_7^3$, $x_0x_1x_2x_3x_4^3x_5^3x_6^3x_7^3$, $x_1^2x_3^4x_5^2x_7^4x_2^2x_6^2$, $x_1^2x_3^2x_5^4x_7^4x_4^2x_6^2$, $x_2^2x_3^2x_6^4x_7^4x_4^2x_5^2$.

- *In* $S_{\rho-\alpha_1}$:

$x_0^4x_1^4x_2^4x_3^4$, $x_0^4x_1^4x_4^4x_5^4$, $x_0^4x_2^4x_4^4x_6^4$, $x_1^4x_3^4x_5^4x_7^4$, $x_2^4x_3^4x_6^4x_7^4$, $x_4^4x_5^4x_6^4x_7^4$, $x_0^4x_1^4x_2^2x_3^2x_4^2x_5^2$, $x_0^4x_1^2x_2^4x_3^2x_4^2x_6^2$, $x_0^3x_1^3x_2^3x_3^3x_4x_5x_6x_7$, $x_0^2x_1^4x_2^2x_3^4x_5^2x_7^2$, $x_0^2x_1^2x_2^4x_3^4x_6^2x_7^2$, $x_0^4x_1^2x_4^4x_5^2x_2^2x_6^2$, $x_0^3x_1^3x_4^3x_5^3x_2x_3x_6x_7$, $x_0^2x_1^4x_4^2x_5^4x_3^2x_7^2$, $x_0^3x_2^3x_4^3x_6^3x_1x_3x_5x_7$, $x_0^2x_2^4x_4^2x_6^4x_3^2x_7^2$, $x_0x_1^3x_2x_3^3x_4x_5^3x_6x_7^3$, $x_0x_1x_2^3x_3^3x_4x_5x_6^3x_7^3$.

- *In* $S_{\rho-\alpha_2}$:

$x_0^4x_1^4x_2^4x_3^4$, $x_0^4x_1^4x_4^4x_5^4$, $x_0^4x_2^4x_4^4x_6^4$, $x_1^4x_3^4x_5^4x_7^4$, $x_2^4x_3^4x_6^4x_7^4$, $x_4^4x_5^4x_6^4x_7^4$, $x_0^4x_1^4x_2^2x_3^2x_4^2x_5^2$, $x_0^4x_1^2x_2^4x_3^2x_4^2x_6^2$, $x_0^3x_1^3x_2^3x_3^3x_4x_5x_6x_7$, $x_0^2x_1^4x_2^2x_3^4x_5^2x_7^2$, $x_0^4x_1^2x_4^4x_5^2x_2^2x_6^2$, $x_0^3x_1^3x_4^3x_5^3x_2x_3x_6x_7$.

- *In* $S_{\rho-\alpha_3}$:

$x_0^4x_1^4x_2^4x_3^4$, $x_0^4x_1^4x_4^4x_5^4$, $x_0^4x_2^2x_4^4x_6^4$, $x_1^4x_3^4x_5^4x_7^4$, $x_2^4x_3^4x_6^4x_7^4$, $x_4^4x_5^4x_6^4x_7^4$, $x_0^4x_1^4x_2^2x_3^2x_4^2x_5^2$.

Again in this case, for any polynomial P of critical degree, we have that $\textbf{Residue}_F(P) = \pm\frac{\det(\tilde{\mathbb{M}}_P)}{\det(\tilde{\mathbb{M}})}$.

4. Toric Subresultants in the critical degree

In this section we define the toric subresultant of a monomial $h \in S_\rho$ and show that this is precisely the numerator of the residue of h. To set things up we must construct two complexes of free $\mathbb{A}$ modules wich we will call the resultant and subresultant complexes respectively. Along the way we will prove Theorem 3.1 from the previous section. The approach will be to use Weyman's complex [GKZ, Section 3.4E] to pass from an exact sequence of sheaves to a generically exact complex of free modules.

Proof of Theorem 3.1. Note that, as $\textbf{Residue}_F(\Delta_{\bar{\sigma}}) = \pm 1$, $\Delta_{\bar{\sigma}}$ is not in the ideal $\langle F_0, \ldots, F_n\rangle$. Hence the surjectivity of ϕ and the fact that $(S_F)_\rho$ has codimension 1 are equivalent.

Let F be our standard generic ample system. The polynomials F_i are sections of sheaves $\mathcal{L}_i := \mathcal{O}(\alpha_i)$ on X (with coefficients in $\mathbb{A}$). Given any subset I of $\{0, \ldots, n\}$, let $\alpha_I = \sum_{i\in I}\alpha_i$. We get a corresponding (dual) Koszul complex of sheaves:

$$0 \to \mathcal{O}(-\sum \alpha_i) \to \cdots \to \bigoplus_i \mathcal{O}(-\alpha_i) \xrightarrow{F} \mathcal{O}_X \to 0. \tag{5}$$

If we now tensor this complex with the sheaf $\mathcal{M} := \mathcal{O}(\rho) = \mathcal{O}(\sum \alpha_i - \beta_0)$ we get a new complex:

$$0 \to \mathcal{O}(-\beta_0) \to \cdots \to \bigoplus_i \mathcal{O}(\rho - \alpha_i) \xrightarrow{F} \mathcal{O}(\rho) \to 0. \tag{6}$$

Since all sheaf Tor groups vanish when one of the factors is locally free, it follows that the complex (6) remains exact even if $\mathcal{M}$ is not locally free. We can therefore apply "Weyman's complex" [GKZ, Chapter 3, Theorem 4.11] which yields a double complex:

$$\begin{aligned} C^{-p,q} &= H^q\left(X, \bigoplus_{0\le i_1\cdots\le i_p\le n} \mathcal{L}^*_{i_1}\otimes\cdots\otimes\mathcal{L}^*_{i_p}\otimes\mathcal{M}\right) \\ &= \bigoplus_{|I|=n+1-p} H^q(X, \mathcal{O}(\alpha_I - \beta_0)). \end{aligned}$$

The corresponding total complex is generically exact with differentials depending polynomially on the coefficients of the F_i; therefore we can view this as a generically exact complex of free $\mathbb{A}$-modules. By the

toric version of Kodaira vanishing, see [Mus], all cohomology terms in the complex vanish except when $q = 0$ or when $p = n+1$ and $q = n$. Note that $H^0(X, \mathcal{O}(\alpha_I - \beta_0)) = S_{\alpha_I - \beta_0}$ and the differentials between these terms are just those from the Koszul complex on S determined by $F_0, \dots, F_n$. Also the only non-vanishing higher cohomology term is $H^n(X, \mathcal{O}(-\beta_0)) \cong \mathbb{C}$, which will correspond to a rank 1 free $\mathbb{A}$-module. The last differential $\phi : C^{-1} \to C^0$ can therefore be chosen to be the map (4) from Section 3.

$$\begin{array}{rccc} \phi : & S_{\rho-\alpha_0} \oplus \cdots \oplus S_{\rho-\alpha_n} \oplus \mathbb{A} & \to & S_\rho \\ & (G_0, \dots, G_n, c) & \mapsto & \sum_{i=0}^n G_i F_i + c\,\Delta_{\tilde{\sigma}}. \end{array}$$

This map is generically surjective and therefore we have proven 3.1. $\square$

Definition 4.1. *The* **resultant complex** *is the complex of free $\mathbb{A}$-modules constructed above. Namely*

$$0 \to S_{-\beta_0} \to \cdots \to \bigoplus_i S_{\rho-\alpha_i} \oplus \mathbb{A} \xrightarrow{\phi} S_\rho \to 0 \tag{7}$$

where the map ϕ is as above.

Let $\mathcal{A}_i := \{a \in \mathbb{N}^s : \deg(a) = \alpha_i\}$, and ℓ the index of the lattice spanned by $\cup \mathcal{A}_i$ in $\mathbb{Z}^n$.

Proposition 4.2. *The determinant of the resultant complex (7) with respect to the monomial bases is*

$$c \cdot \mathrm{res}_{\alpha_0,\dots,\alpha_n}(F_0, \dots, F_n)^\ell$$

for some constant $c \in \mathbb{Q}$.

Proof. In the case when $\ell = 1$, so the $\mathcal{A}_i$ span $\mathbb{Z}^n$, this is a consequence of Theorem 4.11 in [GKZ, Chapter 3.4E]. For the general case we note that the determinant still vanishes if and only if the resultant is 0, and the degree with respect to the coefficients of any F_i is still the mixed volume of all of the other supports with respect to the given lattice $\mathbb{Z}^n$. The degree of the resultant, on the other hand, is $\frac{1}{\ell}$ of this mixed volume [PS]. $\square$

Conjecture 4.3. *The constant $c = \pm 1$.*

The following corollary also appears, under weaker hypothesis on F, in [CoD].

Corollary 4.4. *The complex below of free $\mathbb{A}$-modules is generically exact everywhere but at the last step.*

$$0 \to S_{-\beta_0} \to \cdots \to \bigoplus_i S_{\rho-\alpha_i} \xrightarrow{F} S_\rho. \tag{8}$$

There are two ways to enforce exactness at the last stage of the complex. The resultant complex does this by enlarging the second to last module. The second way to get an exact complex out of (8) is to corestrict the last map to a smaller target. Pick a monomial $h \in S_\rho$ and define $S_\rho/h := \mathbb{A}\langle x^a, \deg(a) = \rho, x^a \neq h\rangle$.

Definition 4.5. *The* **subresultant complex** *with respect to h is the complex*

$$0 \to S_{-\beta_0} \to \cdots \to \bigoplus_i S_{\rho-\alpha_i} \xrightarrow{F_h} S_\rho/h \to 0 \tag{9}$$

Here F_h is the multiplication map of the F_i corestricted to S_ρ/h.

Proposition 4.6. *If h does not belong to the ideal $\langle F_0, \ldots, F_n\rangle$, with coefficients in $Q(\mathbb{A})$, then the complex (9) is generically exact.*

Proof. This is an immediate consequence of Theorem 3.1. □

Now, as the homology with coefficients in $\mathbb{A}$ vanishes for $p > 0$, the determinant of the complex with respect to the monomial bases is an element of $\mathbb{A}$ ([GKZ, Appendix A]) and may be computed as the gcd of the maximal minors of F_h. So we can define:

Definition 4.7. *If h generates $S_{F\rho}$, then let*

$$\mathcal{S}_h := \det(\textit{complex}\,(9)) \in \mathbb{A},$$

where the determinant is taken with respect to the monomial bases of $\mathbb{K}$. If h belongs to ideal generated by $F_0, \ldots, F_n$, then we set $\mathcal{S}_h := 0$. The polynomial $\mathcal{S}_h$ is well-defined, up to a sign, and is called the h-subresultant of the family (1).

Proposition 4.8.

(1) *For each i and each monomials h of degree ρ, $\mathcal{S}_h$ is homogeneous in the coefficients of F_i. If it is not identically zero, it has total degree equal to $\sum_i MV(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n) - 1$. Here MV is the mixed volume of the polytopes corresponding to the given ample degrees.*

(2) *Let $\mathbf{k}$ be a field of characteristic zero. For every specialization of the coefficients of F_i in $\mathbf{k}$, we have that $\mathcal{S}_h \neq 0$ if and only if*

$$\mathbf{k}\langle h\rangle + \langle F_0, \ldots F_n\rangle_\rho = \mathbf{k}[x_0, \ldots, x_{s-1}]_\rho.$$

Proof. The second statement is a straightforward consequence of the definition of $\mathcal{S}_h$ as the determinant of the complex (9). The first part follows by comparing the determinant of (9) and the determinant of

the resultant complex (7), whose degree in the coefficients of f_i equals $MV(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$. □

5. Residues, Resultants and Subresultants

We are now ready for our second main theorem.

Theorem 5.1.

$$\textbf{Residue}_F\,(h) = \pm \frac{\mathcal{S}_h}{c\,.\,\mathrm{res}_{\alpha_0,\ldots,\alpha_n}(F_0, \ldots, F_n)^{\ell}}.$$

Proof. Using the result of Theorem 3.2:

$$\textbf{Residue}_F\,(h) = \pm \frac{\det(\tilde{\mathbb{M}}_h)}{\det(\tilde{\mathbb{M}})} = \pm \frac{\delta_1 \mathcal{S}_h}{\delta_1\, c\, \mathrm{res}_{\alpha_0,\ldots,\alpha_n}(F_0, \ldots, F_n)^{\ell}}.$$

This is due to the fact that the extraneous factor δ_1 for a maximal minor in the resultant complex and for the corresponding maximal minor in any subresultant complex are the same. The result now follows from the definition of the subresultant and Proposition 4.2. □

Corollary 5.2. *We get the following factorization in* $\mathbb{A}$:

$$\mathcal{S}_h = \mathrm{res}_{\alpha_0,\ldots,\alpha_n}(F_0, \ldots, F_n)^{\ell-1}\, P_h,$$

where P_h *is a polynomial which is not a factor of the resultant.*

Proof. In [CDS2, Theorem 1.4] it is shown that the residue is a rational function whose denominator is $\mathrm{res}_{\alpha_0,\ldots,\alpha_n}(F_0, \ldots, F_n)$. While their proof is in the generalized unmixed setting, it carries over to our setting as well, since it relies only on the representation of the toric residue as a sum of local residues, [CCD, Theorem 0.4]. Counting degrees, it turns out that P_h has degree in the coefficients of f_i one less than the sparse resultant. So, it cannot be a factor of it. □

Theorem 5.1 may be regarded as a generalization of Jouanolou's results in the dense case. Suppose that $F_0, \ldots, F_n$ are generic homogeneous polynomials of respective degrees $d_0, \ldots, d_n$. In [J1, (2.9.6)], a linear function $\omega : (S/\langle F_0, \ldots, F_n\rangle)_\rho \to \mathbb{A}$ is defined by setting $\omega(\Delta_{\bar{\sigma}}) := res_{d_0,\ldots,d_n}(F_0, \ldots, F_n)$. Hence, $\omega(h)$ may be regarded as the numerator of the residue of h. Several properties of this morphism are studied in [J2, J3]. In [J3, Corollaire 3.9.7.7], it is shown that $\omega(h)$ may be computed as a quotient of two determinants. Comparing this quotient with Chardin's recipe for computing the subresultant as a quotient of two determinants ([Cha2]) we get that, if h is a monomial then $\omega(h)$ is the classical subresultant of the set $\{h\}$ with respect to $F_0, \ldots, F_n$.

Example 5.3. *We present here an example where $\ell > 1$. Let P be the simplex in $\mathbb{R}^3$ which is the convex hull of $(0,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(3,1,1)$. In this case, $\ell = 3$ and S is a ring of polynomials in 4 variables. Let $\alpha = P \cap \mathbb{Z}^3$, and consider the following four generic polynomials in S_α :*

$$F_i := a_i x_1^3 + b_i x_2^3 + c_i x_3^3 + d_i x_4^3, \ i = 0,1,2,3.$$

Δ_σ of this system equals $x_1^2x_2^2x_3^2x_4^2$ times the determinant of

$$D := \begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}.$$

Also, it is easy to see that $\operatorname{res}_{\alpha,\alpha,\alpha,\alpha}(F_0,F_1,F_2,F_3) = \det(D)$. The matrix of the last morphism of the complex (4) has size 33×21 in this case. A nonzero maximal minor of this matrix is

$$\left(\begin{array}{ccccccccccccccccccccc}
0 & a_1 & b_1 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_2 & b_2 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_3 & b_3 & c_3 & d_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_4 & b_4 & c_4 & d_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & b_1 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_2 & b_2 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_3 & b_3 & c_3 & d_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & 0 & d_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 \\
0 & 0 & 0 & a_2 & 0 & 0 & d_2 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & b_3 & c_3 & d_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & b_4 & c_4 & d_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & d_1 & 0 & 0 & c_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & d_2 & 0 & 0 & c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & b_3 & d_3 & 0 & 0 & c_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & b_1 & c_1 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & b_2 & c_2 & d_2 & 0 & 0 & 0 \\
0 & 0 & a_1 & 0 & 0 & d_1 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & d_1 & 0 & c_1 & 0 & 0 & 0 & b_1 & 0 \\
\det(D) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

Its determinant equals

$$\det(D)^3\,(c_2b_1 - b_2c_1)^2(d_3c_2b_1 - b_2c_1d_3 - d_1b_3c_2 - c_3d_2b_1 + c_3d_1b_2 + d_2b_3c_1)^2$$

so we have that

$$\delta_1 = (c_2b_1 - b_2c_1)^2(d_3c_2b_1 - b_2c_1d_3 - d_1b_3c_2 - c_3d_2b_1 + c_3d_1b_2 + d_2b_3c_1)^2.$$

The first column of this matrix is indexed by $x_1^2x_2^2x_3^2x_4^2$. It is easy to see that, for every monomial h of critical degree, $\mathcal{S}_h = 0$ unless $h = x_1^2x_2^2x_3^2x_4^2$. In this case, $\mathcal{S}_{x_1^2x_2^2x_3^2x_4^2} = \det(D)^2$ and hence $P_{x_1^2x_2^2x_3^2x_4^2} = \pm 1$.

This can be explained as follows: every monomial of critical degree is a multiple of x_i^3 for at least one $i = 0, \dots, 3$ except $x_1^2x_2^2x_3^2x_4^2$. An easy

consequence of Cramer's rule is that every monomial which is multiple of x_i^3 is in the ideal generated by the generic F_i's. This is why all except one of the subresultants are identically zero.

Conjecture 5.4. *If P_h is not identically zero, then it is an irreducible element of $\mathbb{A}$. In particular, when $\ell = 1$, every subresultant $\mathcal{S}_h$ is irreducible.*

6. Computing global residues "Á la Macaulay"

In this section, we will review the toric algorithm of [CD] for computing global residues by means of toric residues (see also [CDS2]). As a straightforward consequence of their algorithm and our results, we get a quotient formula for computing global residues. In the dense case, we recover the quotient type formula given by Macaulay in [Mac] for computing the global residue of $x_1^{d_1-1}x_2^{d_2-1}\ldots x_n^{d_n-1}$ with respect to a generic family of polynomials of degrees $d_1, \ldots, d_n$.

Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ subsets of $\mathbb{Z}^n$, and consider n Laurent polynomials in n variables $t_1, \ldots, t_n$ having support in $\mathbf{A}_1, \ldots, \mathbf{A}_n$ respectively:

$$f_j \quad = \sum_{m \in \mathbf{A}_j} u_{jm} \cdot t^m \; j = 1, \ldots, n.$$

Let V be the set of common zeros of $f_1, \ldots, f_n$ in the torus $T = (\mathbb{C}^*)^n$. If V is finite and all its roots are simple, then for any Laurent polynomial $q \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the *global residue* of the differential form

$$\phi_q \quad = \quad \frac{q}{f_1 \cdots f_n} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n},$$

is defined as $\sum_{\xi \in V} \frac{q(\xi)}{J^T(f)(\xi)}$, where $J^T(f)$ denotes the *affine toric Jacobian*

$$J^T(f) \quad := \quad \det\big(t_k \frac{\partial f_j}{\partial t_k}\big)_{1 \leq j,k \leq n}.$$

Global residues are basic invariants of multivariate polynomial systems (see [CD, CDS2] and the references therein).

The link between toric and global residues is given in [CD, Theorem 4]:

Theorem 6.1. *Let $f_1, \ldots, f_n \in \mathbb{C}[t_1^{\pm 1} \ldots, t_n^{\pm 1}]$ be Laurent polynomials having a finite number of zeroes in T, and g another Laurent monomial. Then there is a projective toric variety X with homogeneous coordinate ring S_X and a homogeneous element $F_0 \in S_X$ such that*

(1) *There is an "homogenization rule" which assigns to every f_i a homogeneous polynomial $F_i \in S_X$, $i = 1, \ldots, n$;*
(2) *the family $F_0, \ldots, F_n$ has no zeros in X;*
(3) *There is another homogeneous monomial $G \in S_X$ such that*

$$\textbf{Global Residue}_f\,(g) = \textbf{Residue}_F\,(G).$$

As an immediate consequence, we get also a quotient formula for computing global residues, as the following example shows:

Example 6.2. *This example has already appeared in the introduction of* [CDS2]. *We want to compute the global residue of $g(t) := t_1^3 t_2^2$ with respect to the generic system*

$$\begin{array}{rcl} f_1 & = & a_0t_1^2 + a_1t_1t_2 + a_2t_2^2 + a_3t_1 + a_4t_2 + a_5\,, \\ f_2 & = & b_0t_1^2 + b_1t_1t_2 + b_2t_2^2 + b_3t_1 + b_4t_2 + b_5\,. \end{array}$$

Applying the algorithm of [CD] *we get that, in this case, the corresponding toric variety X is $\mathbb{P}^2$ with the standard homogeneous coordinates $S_X = \mathbb{C}[x_0, x_1, x_2]$. The "homogeneous" polynomials are*

$$\begin{array}{l} F_0(x_0, x_1, x_2) := x_0^2 \\ F_1(x_0, x_1, x_2) := a_0x_1^2 + a_1x_1x_2 + a_2x_2^2 + a_3x_1x_0 + a_4x_2x_0 + a_5x_0^2 \\ F_2(x_0, x_1, x_2) := b_0x_1^2 + b_1x_1x_2 + b_2x_2^2 + b_3x_1x_0 + b_4x_2x_0 + b_5x_0^2, \end{array}$$

and $G = x_1^2x_2$.

The global residue may be computed then as the toric residue of G with respect to F_0, F_1, F_2. Using our methods, it turns out that the matrix $\mathbb{M}$ of Theorem 3.2 is square. Computing it explicitly, we get

$$\mathbb{M} = \begin{array}{c} \begin{array}{cccccccccc} x_0x_1^2 & x_0x_1x_2 & x_0x_2^2 & x_0^2x_1 & x_0^2x_2 & x_0^3 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\ \left(\begin{array}{cccccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & \mathbf{0} & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{0} & 0 & 0 \\ a_3 & a_4 & 0 & a_5 & 0 & 0 & a_0 & \mathbf{a_1} & a_2 & 0 \\ b_3 & b_4 & 0 & b_5 & 0 & 0 & b_0 & \mathbf{b_1} & b_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & a_3 & a_4 & 0 & a_5 & 0 & 0 & \mathbf{a_0} & a_1 & a_2 \\ 0 & b_3 & b_4 & 0 & b_5 & 0 & 0 & \mathbf{b_0} & b_1 & b_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 \\ \mathbf{j_1} & \mathbf{j_2} & \mathbf{j_3} & \mathbf{j_4} & \mathbf{j_5} & \mathbf{j_6} & \mathbf{j_7} & \mathbf{j_8} & \mathbf{j_9} & \mathbf{j_{10}} \end{array}\right) \end{array},$$

where the $\mathbf{j_k}$ are the coefficients of $\Delta_{\bar{\sigma}}$. The matrix $\tilde{\mathbb{M}}_h$ is made by deleting in $\mathbb{M}$ the eighth row and the last column. Hence, we have in

virtue of Theorem 3.2

$$\mathbf{Residue}_F(h) = \pm\frac{\det(\tilde{\mathbb{M}}_h)}{\det(\mathbb{M})} = \pm\frac{\det(\tilde{\mathbb{M}}_h)}{\mathrm{res}_{2,2,2}(F_0,F_1,F_2)},$$

and one can check that $\det(\tilde{\mathbb{M}}_h)$ *is the polynomial* P_{32} *of the introduction of* [CDS2].

We close this section by showing that the method for computing global residues as a quotient of two determinants presented here, may be regarded as a generalization of a formula given by Macaulay in the classical case. In order to follow his notation, let $F_1(x_0,\ldots,x_n),\ldots,$ $F_n(x_0,\ldots,x_n)$ be generic homogeneous polynomials of respective degrees $d_1,\ldots,d_n$, and set $f_i := F_i(1,x_1,\ldots,x_n)$. Let J be the affine Jacobian of the f_i, i.e. $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$, and

$$V(f_1,\ldots,f_n) := \{\xi_1,\ldots,\xi_{d_1\ldots d_n}\} \subset \overline{Q(\mathbb{A})}$$

be the variety defined by the common zeroes of the f_i in the algebraic closure of $Q(\mathbb{A})$. We denote with $\mathbf{m}$ the monomial $x_1^{d_1-1}\ldots x_n^{d_n-1}$.

From display (13) in [Mac] and following his notation, we have

$$\sum_{j=1}^{d_1\ldots d_n}\frac{\mathbf{m}(\xi_j)}{J(\xi_j)} = \pm\frac{R(n,t_n-1)}{R(n,t_n)}, \tag{10}$$

where

- $R(n,t_n)$ is the resultant of $F_1(0,x_1,\ldots,x_n),\ldots,F_n(0,x_1,\ldots,x_n)$,
- $R(n,t_n-1)$ is the subresultant of the monomial $\mathbf{m}$ with respect to $F_1(0,x_1,\ldots,x_n),\ldots,F_n(0,x_1,\ldots,x_n)$.

Comparing the left hand side of (10) with the definition of the definition given above, we have that (10) is actually the global residue of $x_1^{d_1}\ldots x_n^{d_n}$ with respect to $f_1,\ldots,f_n$. Applying the toric algorithm of [CD] we get the following:

- $X = \mathbb{P}^n$, $S_X = \mathbb{C}[x_0,\ldots,x_n]$ with homogenization given by total degree;
- $F_0 = x_0$, and if $g = x_1^{d_1}\ldots x_n^{d_n}$, then $G = \mathbf{m}$.

Denote with res the homogeneous resultant for a family of $n+1$ homogeneous polynomials in $n+1$ variables of degrees $1,d_1,\ldots d_n$. Then, it turns out that (10) equals $\mathbf{Residue}_F(\mathbf{m})$. Applying Theorem 5.1, we can write $\mathbf{Residue}_F(\mathbf{m})$ as $\pm\frac{\mathcal{S}_{\mathbf{m}}}{\mathrm{res}(x_0,F_1,\ldots,F_n)}$. Now, specializing a generic F_0 to x_0 and applying [Cha2, Lemma 1], we get that $\mathcal{S}_{\mathbf{m}} \mapsto R(n,t_n-1)$ and res $\mapsto R(n,t_n)$.

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References

[Bez] É. Bézout. *Théorie générale des équations algébriques.* Ph.-D. Pierres: Paris.

[BC] V. Batyrev and D. Cox. *On the Hodge structure of projective hypersurfaces in toric varieties,* Duke J. Math. 75 (1994), 293-338

[BM] V. Batyrev and D. Cox. *Mixed toric residues and Calabi-Yau complete intersections.* math.AG/0206057

[Cha1] M. Chardin. *Multivariate subresultants.* J. Pure Appl. Algebra 101, No.2, 129-138 (1995).

[Cha2] M. Chardin. *Formules à la Macaulay pour les sous-résultants en plusieurs variables.* C. R. Acad. Sci. Paris Sér. I Math. 319 (1994), no. 5, 433–436.

[CCD] E. Cattani, D. Cox, and A. Dickenstein. *Residues in toric varieties.* Compositio Math. 108 (1997), no. 1, 35–76.

[CoD] D. Cox and A. Dickenstein. *Vanishing and codimension one theorems on complete toric varieties* In progress, personal communication

[CD] E. Cattani and A. Dickenstein. *A global view of residues in the torus.* Algorithms for algebra (Eindhoven, 1996). J. Pure Appl. Algebra 117/118 (1997), 119–144.

[CDS1] E. Cattani, A. Dickenstein, and B. Sturmfels. *Computing multidimensional residues.* Algorithms in algebraic geometry and applications (Santander, 1994), 135–164, Progr. Math., 143, Birkhäuser, Basel, 1996.

[CDS2] E. Cattani, A. Dickenstein, and B. Sturmfels. *Residues and resultants.* J. Math. Sci. Univ. Tokyo 5 (1998), no. 1, 119–148.

[Cox1] D. Cox. *The homogeneous coordinate ring of a toric variety.* J. Algebr. Geom., 4, 17–50 (1995).

[Cox2] D. Cox. *Toric residues.* Ark. Mat. 34 (1996), no. 1, 73–96.

[CLO1] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms. An introduction to computational algebraic geometry and commutative algebra.* Undergraduate Texts in Mathematics. New York, NY: Springer. xiii, 1996.

[CLO2] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry.* Graduate Texts in Mathematics, 185. Springer-Verlag, New York, 1998.

[Ful] W. Fulton *Introduction to Toric Varieties.* Ann. of Math. Studies, no. 131, Princeton NJ, Princeton University Press (1993).

[GH] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* John Wiley & Sons, New York, 1978.

[GK] O.A. Gelfond, and A.G. Khovanskii. *Toric geometry and Grothendieck residues.* Mosc. Math. J. 2 (2002), no. 1, 99–112, 199. 14M25

[GKZ] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[J1] J.P. Jouanolou. *Singularités rationnelles du résultant.* Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), pp. 183–213, Lecture Notes in Math., 732, Springer, Berlin, 1979.

[J2] J.P. Jouanolou. *Aspects invariants de l'limination.* Adv. Math. 114 (1995), no. 1, 1–174.

[J3] J.P. Jouanolou. *Formes d'inertie et résultant: un formulaire.* (French) Adv. Math. 126 (1997), no. 2, 119–250.

[Kho] A.G. Khovanskii. *Newton polyhedra and toroidal varieties.* Russian Math. Surveys 33 (1978), 237–238.

[Mac] F. Macaulay. *Some formulae in elimination*. Proc. London Math. Soc. 1 **33**, 3–27, 1902.

[Mus] M. Mustaţă *Vanishing theorems on toric varieties*. Tohoku Math. J.(2) **54** (2002), no. 3, 451–470.

[PS] P. Pedersen and B. Sturmfels, *Product formulas for resultants and Chow forms*. Mathematische Zeitschrift **214** (1993), 377-396.

[Sop] I. Soprounov. *Residues and tame symbols in toric geometry.* math.AG/0203114.

[Sza] A. Szanto. *Multivariate subresultants using Jouanolou's resultant matrices.* Preprint.

[VdW] B. L. van der Waerden. *Modern Algebra.* 3^{rd} edn. New York, F. Ungar Publishing Co., 1950.

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